2.1: Population Models

We saw in Section 1.4 that if we wish to model a population growth with constant birth and death rates then we must solve the differential equation

$$\frac{dP}{dt} = kP, \tag{1}$$

which has solution $P(t) = P_0 e^{kt}$. However, in real-world situations a constant birth and death rate is almost certainly unreasonable in many situations. Consider, for instance, the case where we want to take into account the fact that there is only a finite amount of water in the world. In this, and many other cases, we need a more general population model; i.e. a model which takes into account non-constant birth and death rates. If we let

$$\begin{aligned} \beta(t) &= \# \text{ of births per unit of population per unit of time at time } t. \\ \delta(t) &= \# \text{ of delta per unit of population per unit of time at time } t., \end{aligned} \tag{2}$$

then we arrive at the **general population** equation

$$\frac{dP}{dt} = [\beta(t) - \delta(t)]P(t) \quad \text{or} \quad \frac{dP}{dt} = (\beta - \delta)P. \tag{3}$$

Example 1. Suppose that an alligator population numbers 100 initially, and that its death rate $\delta = 0$. If the birth rate if $\beta = (0.0005)P$, solve the differential equation (3).

Question 1. The above example is not very realistic for an extended period of time. Why?

Definition 1. A more realistic situation where the limiting population is bounded (see Section 1.3, Example 3) is given by the **logistic equation**

$$\frac{dP}{dt} = (\beta_0 - \beta_1 P - \delta_0)P \quad \text{or} \quad \frac{dP}{dt} = aP - bP^2, \tag{4}$$

where $a = \beta_0 - \delta_0$ and $b = \beta_1$.

Example 2. Suppose that the birth rate for a given population is $\beta = 5 - (0.005)P$ and the death rate is $\delta = 3$. Solve the logistic equation (4).

Finally, we wish to rewrite the logistic equation (4) one last time to emphasize another quantity. We get

$$\frac{dP}{dt} = kP(M - P), \tag{5}$$

where $k = b = \beta_1$ and $M = \frac{a}{b} = \frac{\beta_0 - \delta_0}{\beta_1}$ is known as the **carrying capacity.** We can use the method of separable equations to show that the solution to (5), with the initial condition $P(0) = P_0$, is given by

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}. \tag{6}$$

Question 2. What happens to $P(t)$ in the limit as $t \to \infty$? Consider the cases $P_0 > M$, $P_0 < M$ and $P_0 = M$ separately.

Exercise 1. The U.S. population in 1800 was 5.308 million and in 1900 was 76.212 million. Solve for the U.S. population $P(t)$ in years since 1800 first using the natural growth equation (1) and then using the logistic equation (5). When you are done, see Figure 2.1.4 from page 79 of the text for very strong support for the logistic equation as a model for human population.

Homework. 1-5, 9-21 (odd)